LETTER TO THE EDITOR

Surface-to-bulk crossover in directed compact percolation

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Abstract. For directed compact percolation in two dimensions, we conjecture the exact analytical expressions for the percolation probability near a non-conducting wall. The initial 'wet' site is at a varying distance from the wall which is along the oriented 'time' direction. Our results allow an explicit evaluation of the surface-to-bulk crossover in the percolation probability.

Surface properties of spin models near-phase transitions have attracted much theoretical attention [1, 2]. For isotropic percolation models, surface effects have been studied less systematically; several results are available [3-9]. In fact, studies of percolation near surfaces provide tools to describe processes of surface dissolution and corrosion [10].

Surface properties of *directed percolation*, however, were only investigated within the ε-expansion down from $d = 5$ [11]. It would be of interest to obtain analytical results for simplified, low-dimensional versions of directed percolation. Indeed, in this note we conjecture the exact expressions for the percolation probability near a wall in the *compact* directed percolation introduced recently by Essam [12].

Directed compact percolation is defined on the square lattice (figure 1) with the axes rotated 45° with respect to the space and time directions. In figure 1, a typical finite cluster spreading from the initial ($t = 0$) site at $L = 0$ is shown by the full circles. The shaded circles indicate the wall the sites of which act as *empty* or non-conducting in the propagation of connectivity ('wetness'). The coordinate L, defined in figure 1, then provides a convenient measure of the distance from the wall. The connectivity rules are defined [12] to have no gaps at any t-value (provided the cluster is grown from a compact seed at $t = 0$). If two adjacent sites are both wet (both empty) at time t then at time $(t + 1)$ the site between the original two sites (see figure 1) will be wet (empty). In the other two cases (wet-empty or empty-wet) the offspring site is wet with probability p and empty otherwise (probability $1 - p$). Essam [12] calculated several properties of the above model. Specifically, he derived the percolation probability, $P(p)$, for the connectivity to persist to $t = \infty$, starting from a single site at $t = 0$, with no walls,

$$P_\infty(p) = \begin{cases} (2p - 1)/p^2 & \text{for} \quad p \geq \frac{1}{2} \\ 0 & \text{for} \quad p \leq \frac{1}{2}. \end{cases} \tag{1}$$
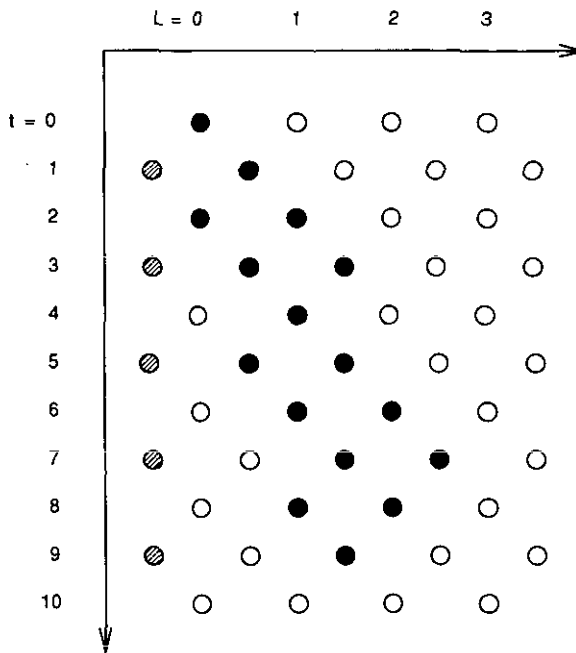
Figure 1. The full circles illustrate a finite compact directed cluster on the square lattice, spreading from the initial site at $L = 0$. The open circles denote the empty lattice sites while the shaded circles represent the wall sites; see text for details.

The 'bulk' ($L = \infty$) percolation transition is at $p_c = \frac{1}{2}$, and the model is obviously dual with respect to $p \rightleftharpoons (1-p)$.

One can also regard the above rule as defining a cellular automaton and consider initial configurations more general than a single continuous segment of wet sites. (This point of view also makes the aforementioned duality property better defined.) The kinetics of such automaton models is characterized by the dynamics of the domain walls separating wet and empty regions. In the most studied and interesting symmetric case ($p = \frac{1}{2}$) the domain wall motion is diffusional, and pairs of walls annihilate in each encounter thus leading to cluster growth. There are no exact results for the cluster size distribution for this model; however, many numerical, exact, and asymptotic results were obtained [12-18] for various diffusive-domain-wall models in 1D, for several interesting properties. Recently, extensions of the Essam model [12] to $d > 1$ were considered [19].

Our consideration will be limited to the original 1D model, with the added complication of the wall at $L = -\frac{1}{2}$ in the notation of figure 1. Note that only the sites at $L > 0$ are updated according to the rules specified above. The wall sites are always empty. The rules for updating at $L = 0$ are defined as follows. Consider the sites at odd times and at $L = \frac{1}{2}$. These sites, if occupied, lead to a wet offspring at $L = 0$ with probability p (e.g. time step $[1 \rightarrow 2]$ in figure 1), whereas the probability of an empty offspring at $L = 0$ is $(1-p)$ (e.g. the time step $[3 \rightarrow 4]$ in figure 1). If the initial site at $L = \frac{1}{2}$ is empty then its offspring at $L = 0$ is always empty.

Our discussion will be focused on the percolation probability $P_L(p)$ which is non-trivial for $p \geq p_c$. The subscript L denotes percolation starting from a single initial site at $L = 0, 1, 2, \ldots$, as defined in figure 1. Note that to the extent that the analogy

with the spin-model surface criticality [1, 2] applies here, the percolation transition near a non-conducting wall must be of the 'ordinary' type; specifically, the percolation threshold should have the same value, p_c, $L = \frac{1}{2}$, as for the bulk.

From the simple form of Essam's results [12], such as equation (1), as well as from the experience with the statistics of directed compact *lattice animals* [20-22], we conjectured that $P_L(p)$ are rational functions of p. We then developed a numerical procedure to obtain these rational expressions for $L = 0, 1, \ldots$, and went on to identify the recursion relation that yielded the result for all integral L values. In the rest of this letter we only outline the numerical part which was mainly used to conjecture the form of the exact expressions. We then analyse the exact results for the crossover scaling behaviour as $p \to p_c^+$ and $L \to \infty$.

In order to outline the numerical approach used, let us define

$$q \equiv 1 - p \tag{2}$$

$$Q \equiv 1 - P. \tag{3}$$

Note that Q is the probability that no infinite cluster is formed, i.e. that the cluster ends at a finite time $t \geq 0$. This quantity is small for small q, and it is natural to consider the high-density expansion of $Q(q)$ in powers of q. There are many ways of calculating such series [23]. The method that we used is rather unconventional and tailored for the particular problem at hand.

We employed the transfer matrix iterations on finite strips, up to times $t = 99$ and for widths up to 16 (in the original lattice distances, see below). The computation proceeded in the following steps. The possible compact clusters for a fixed time were all classified (numerically) and used to label the transfer matrix entries. A table of transition probabilities was then calculated. In order to avoid the difference between the even and odd times (see figure 1), the second, spurious lattice was introduced, shifted half the lattice cell with respect to the original lattice. The list of the allowed cluster states was constructed in such a way as to have the correct clusters and connectivity provided the initial site was on the original lattice. In each time step the transfer matrix was applied on the current state vector (which was initially single-site), and the weight of the fully empty state in the resulting new state vector was calculated as a power series in q. In fact, all the quantities in the calculation were kept as polynomials in q. The details of the connectivity rule at the second wall should typically not matter for the final result. In our case, it was non-conducting.

The part of the resulting series that was unchanged for varying t and lattice width near their maximal values was first examined by the Padé approximant method to try to identify the rational function $Q_L(q)$. However, it was noticed right away that the products $(1 - q)^{2L+3} Q_L(q)$ are in fact polynomials. We then continued our calculations directly for these products and obtained the first few polynomials explicitly.

Before continuing the description of the results, let us comment further on the numerical method employed. Firstly, it is not easily extendible to other problems, notably the full directed percolation model, because the number of the transfer matrix states is exponentially large. For the compact-percolation model, the number of states only grows as $O(L^2)$. With the advantage of the small number of states, the transfer matrix evaluation of the series is much faster than the more conventional cluster-enumeration methods. All our calculations took several CPU hours on the IBM RS/6000 (model 520) workstation. The 'exact' expressions given below, in fact, only represent plausible conjectures based on the computer evaluation and thus are accurate to about

$O(q^{22})$. The consistency of the resulting picture, and the correct limit $L \to \infty$ provide a strong evidence for the validity of the conjectured expressions.

To summarize our results, let us 'translate' them back to $P(p)$. The first few expressions are

$$P_0(p) = (2p-1)^2/p^3 \tag{4}$$

$$P_1(p) = (2p-1)^2(p^2-p+1)/p^5 \tag{5}$$

$$P_2(p) = (2p-1)^2(p^4-2p^3+4p^2-3p+1)/p^7 \tag{6}$$

$$P_3(p) = (2p-1)^2(p^6-3p^5+9p^4-13p^3+11p^2-5p+1)/p^9. \tag{7}$$

These functions are plotted in figure 2 where they are compared with the expression (1). More generally, we found that

$$P_L(p) = \frac{(2p-1)^2}{p^{2L+3}} S_L(p) \tag{8}$$

where the polynomials $S_L(p)$ satisfy the recursion relation (found by inspection)

$$S_{L+1} = p^2 S_L + (1-p)^{2L+1} \qquad \text{with} \quad S_0 = 1. \tag{9}$$

The solution of this recursion finally yields

$$P_L(p) = \frac{2p-1}{p^2} \left[1 - \left(\frac{1-p}{p} \right)^{2L+1} \right]. \tag{10}$$

This result shows an interesting crossover from the bulk behaviour given by the front factor to the surface behaviour with the additional factor suppressing the probability for cluster to spread to infinity. Near p_c this suppression is the most pronounced, with the change in the exponent form the bulk power, 1, to the surface value, 2. Let us denote

$$t = 2p - 1. \tag{11}$$

Then the bulk term in (10) is proportional to t (for small t). However, the L-dependent factor is also proportional to t for any fixed value $L < \infty$.
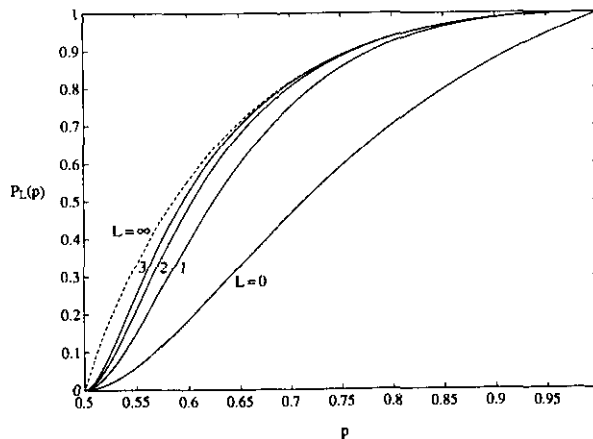

Figure 2. The full lines depict the percolation probabilities $P_L(p)$ for $L = 0, 1, 2, 3$. The broken line gives the $L = \infty$ 'bulk' result, equation (1).

More generally, let us consider the 'scaling limit' of t-small, L-large. We have

$$P_L(p) \approx 4t(1 - e^{-4tL}).$$ (12)

This expression illustrates the crossover scaling: for $t \ll 1$, $L \gg 1$, one can still have the regime of behaviour away from the wall when $L \gg 1/(4t)$ (and $P_L \simeq 4t$), and the wall-dominated regime $L \ll 1/(4t)$ (and $P_L \simeq 16Lt^2$).

Finally, let us comment on some qualitative aspects of directed percolation (compact and non-compact) above the threshold, suggested by our results. In the absence of the wall the connectivity spreads from the initial seed along the infinite cluster which becomes narrow ($\nu_\parallel > \omega_\perp$) as $p \to p_c^+$ and which is mainly confined to the vicinity of the vertical (time) axis passing through the initial seed. Most of the paths of connectivity from the initial seed will cross this axis many times. In the presence of the wall the available paths are limited. The effect is drastic enough to change (actually, double in the present, compact-cluster case) the exponent value for the percolation probability. To our knowledge, the morphology of the infinite percolation cluster, as affected by the wall, has not been investigated thus far.

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